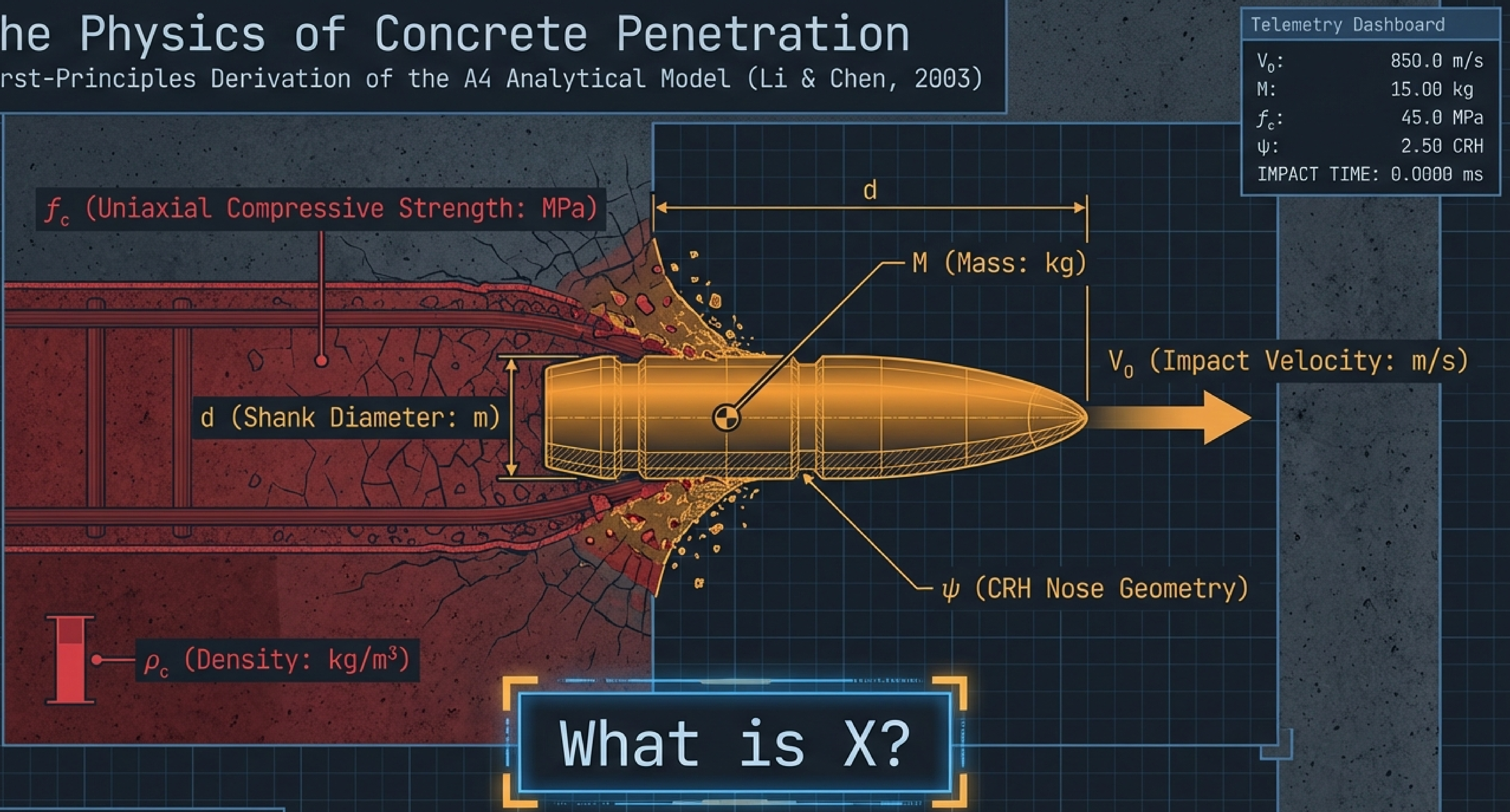
The Physics of Concrete Penetration

First-Principles Derivation of the A4 Analytical Model (Li & Chen, 2003)

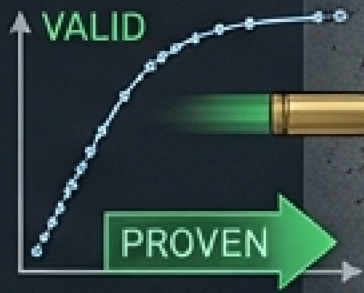


SCHEMATIC ID: BB-ENG-001-A4

The Black Box Problem: Limits of Empirical Formulas

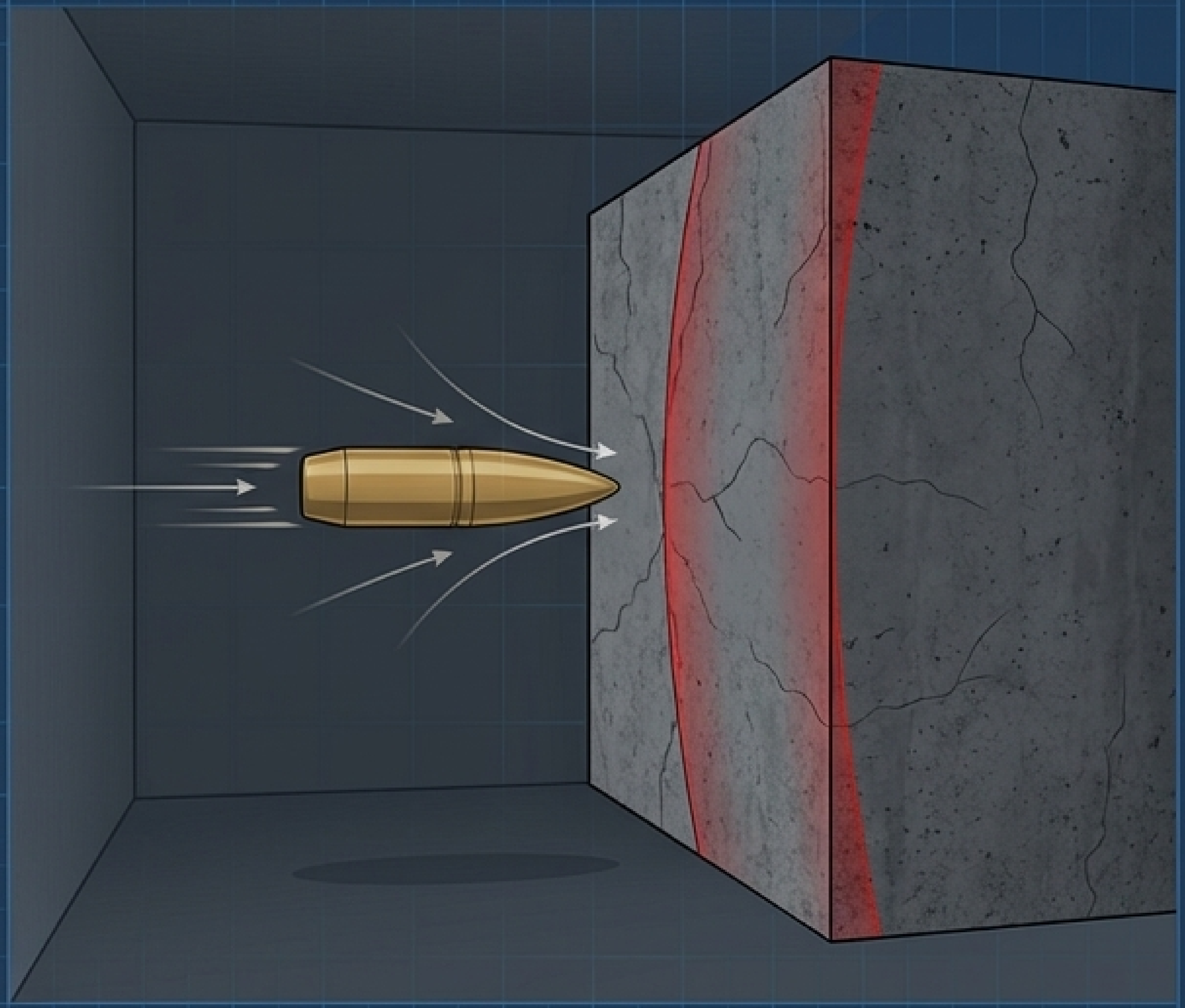
Telemetry Dashboard UI
VELOCITY: 850 m/s | KE: 14.5 MJ
STRESS: 120 MPa | DEPTH: 1.2 m

	Legacy Models (NDRC, ACE)	**The A4 Model** (Li & Chen)
Unit Dependency	Relies on arbitrary coefficients (e.g., 3.8×10^{-5} in SI) that break if units change. $V_N = v_a \left(1 + \frac{v_a}{2.4\, sov_{a}}\right) + h_0 E_{set} \left(\frac{3.8 \times 10^{-5}}{M_U h_0}\right)^{1/2}$	Dimensionally homogeneous. Physics remains identical across any unit system. $E = E_{anv}\frac{1}{a} + v_i(X_2^3 - h_n(z_0^2 + \sigma_0)$
Nose Shape Definition	Discrete, ambiguous assignments (N_N = 0.72 for flat, 1.14 for sharp). Cannot model intermediate or truncated geometries. (0.72 / 1.14 — LIMITED)	Continuous geometric integration (N^*) covering infinite shape variations. $N^* = \int_a N^*$
Range of Validity	Calibrated only for shallow to medium impacts (0.6 < X/d < 2.0). 20-40% errors in deep penetration.	Analytically valid and empirically proven across 3 orders of magnitude (X/d = 0.07 to 92.8).



Empirical formulas fit the data.
Analytical formulas explain the physics.

The Narrative Anchor: “Shot 14”



Telemetry Dashboard UI

STATUS: Pre-Impact

PROJECTILE DATA:

Mass (M):	0.906 kg
Diameter (d):	26.9 mm
Nose Shape:	Ogive (CRH $\psi = 2.0$)
Impact Velocity (V_0):	277 m/s

TARGET DATA:

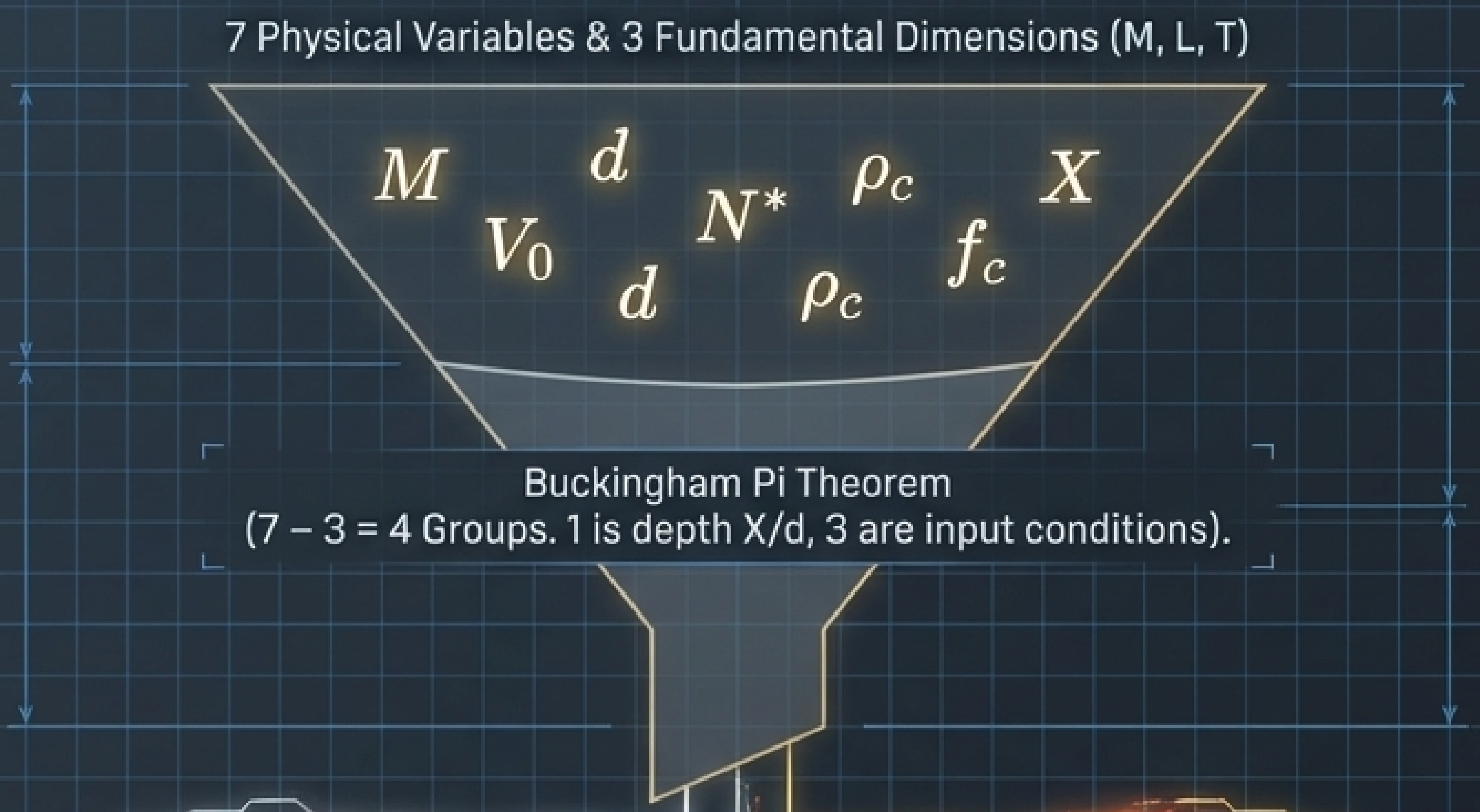
Strength (f_c):	35.2 MPa
Density (ρ_c):	2370 kg/m^3

MEASURED REALITY:

Actual Test Depth (X_{test}): 173 mm

Our goal: Build a mathematical model from scratch that arrives exactly at 173 mm.

Dimensional Analysis: The Buckingham Pi Theorem



1 Impact Factor (I*)

$$I^* = \frac{MV_0^2}{d^3 f_c}$$

(Ratio of kinetic energy to concrete absorption capacity)

2 Mass Ratio (λ)

$$\lambda = \frac{M}{\rho_c d^3}$$

(Bullet mass vs. Target areal density)

3 Nose Factor (N*)

Dimensionless geometric constant for the projectile tip.

SHOT 14 TELEMETRY

I^* Calculated: 101.46

λ Calculated: 19.64

Recombining the Pi Groups: Operational Variables

Synthesis Blocks

Impact Function (I)

$$I = \frac{I^*}{S} = \frac{MV_0^2}{Sd^3 f_c}$$

Divides I* by S (empirical confinement factor) to measure true energy excess.

Synthesis Blocks

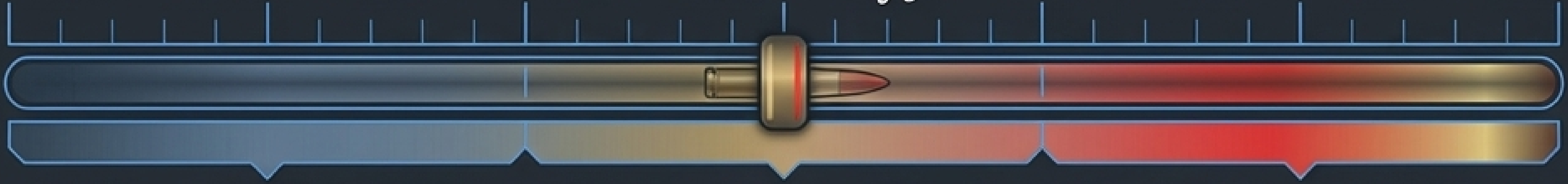
Geometry Function (N)

$$N = \frac{\lambda}{N^*} = \frac{M}{N^* \rho_c d^3}$$

Combines mass and sharpness. Heavy + Sharp = High N.

Johnson's Damage Number (Φ_J)

$$\Phi_J = \frac{I^*}{\lambda} = \frac{\rho_c V_0^2}{f_c}$$



- $\Phi_J \ll 1$: Quasi-static (No damage)
- $\Phi_J \sim 1$: Dynamic (Significant penetration)
- $\Phi_J \gg 1$: Hyper-velocity (Perforation)

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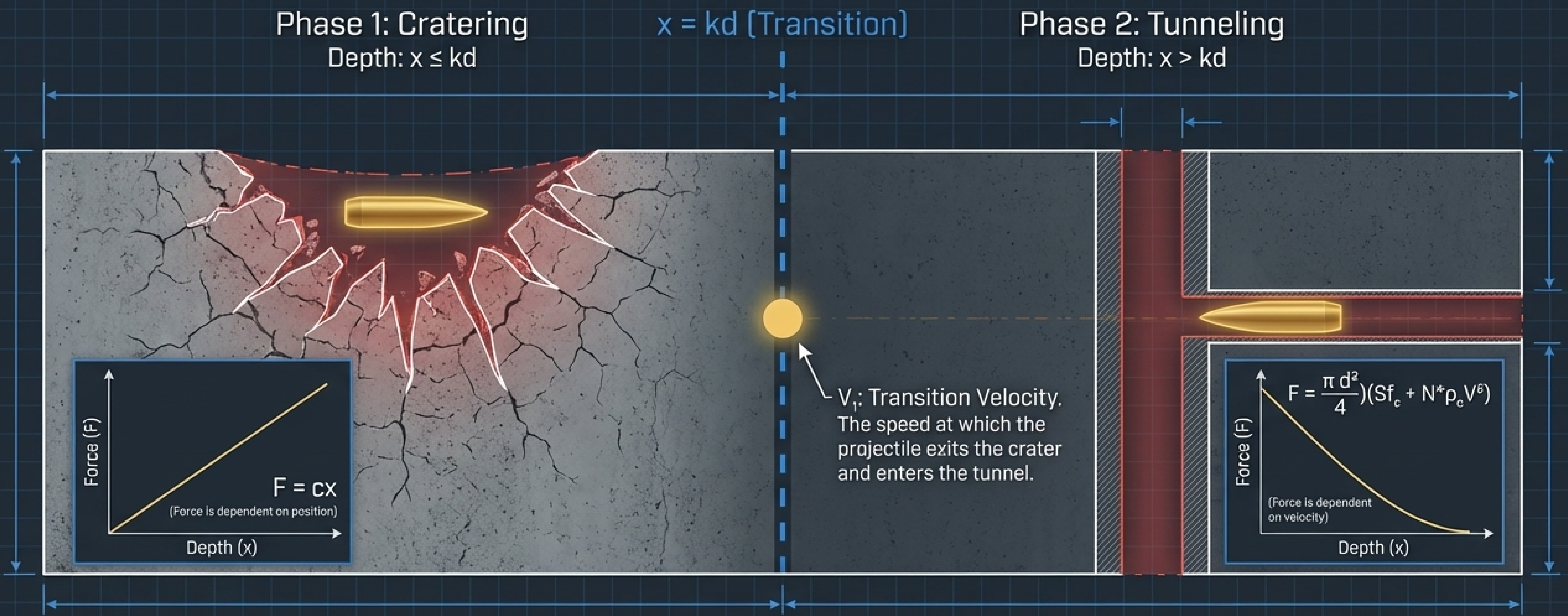
$I = 8.455$

$N = 125.9$

$\Phi_J = 5.17$

The Two Phases of Penetration



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I* Calculated: 101.46

λ Calculated: 19.64

Φ_J = 5.17

Phase 1: Cratering Dynamics & The V_1 Transition

The Equation of Motion: $MV\frac{dV}{dx} = -cx$

[Kinetic Energy at Impact: $\frac{1}{2}MV_0^2$] − [Work Done by Crater: $\frac{1}{2}c\,(kd)^2$] = [Kinetic Energy at Transition: $\frac{1}{2}MV_1^2$]

Deriving V_1^2

$$V_1^2 = \frac{MV_0^2 - \frac{\pi kd^3}{4}Sf_c}{M + \frac{\pi kd^3}{4}N^*\rho_c}$$

Numerator — Initial energy minus static crater absorption.

Denominator — Effective mass (Projectile + Target inertia).

TELEMETRY DASHBOARD

SHOT 14 TELEMETRY

Impact Energy:	34,770 J
Crater Work:	6,810 J
Transition Velocity (V_1):	248.4 m/s
(Projectile survived the crater).	
Crater Depth Limit (kd):	53.8 mm

Phase 2: Tunneling Dynamics & Final Depth (X)

The Differential Equation:

$$M\,V\,\frac{dV}{dx} = -\frac{\pi d^2}{4}\left(S\,f_c + N^{*}\rho_c V^2\right)$$

Force depends entirely on instantaneous velocity V. We integrate from $V = V_1$ down to $V = 0$.

The Dimensional Output (Eq. 35):

$$X = \left[\frac{2M}{\pi d^2 N^{*} \rho_c}\right] \ln\left(1 + \frac{N^{*}\rho_c V_1^2}{S f_c}\right) + kd$$

Length Scale (~2.15 m)

Kinetic Reserve vs Static Resistance

Crater Depth Offset

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Final Velocity (V): 0 m/s
Calculated Depth (X): 167.2 mm

Target hit! But where does this
tunnel force actually come from?
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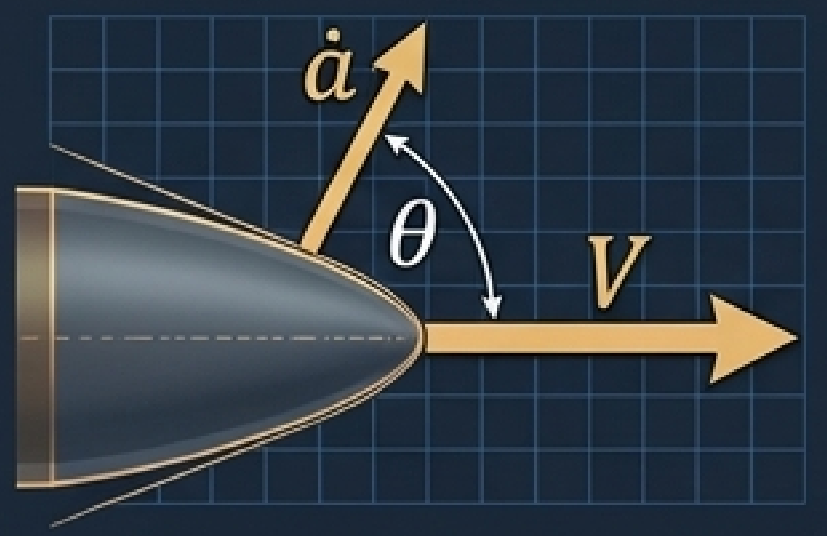
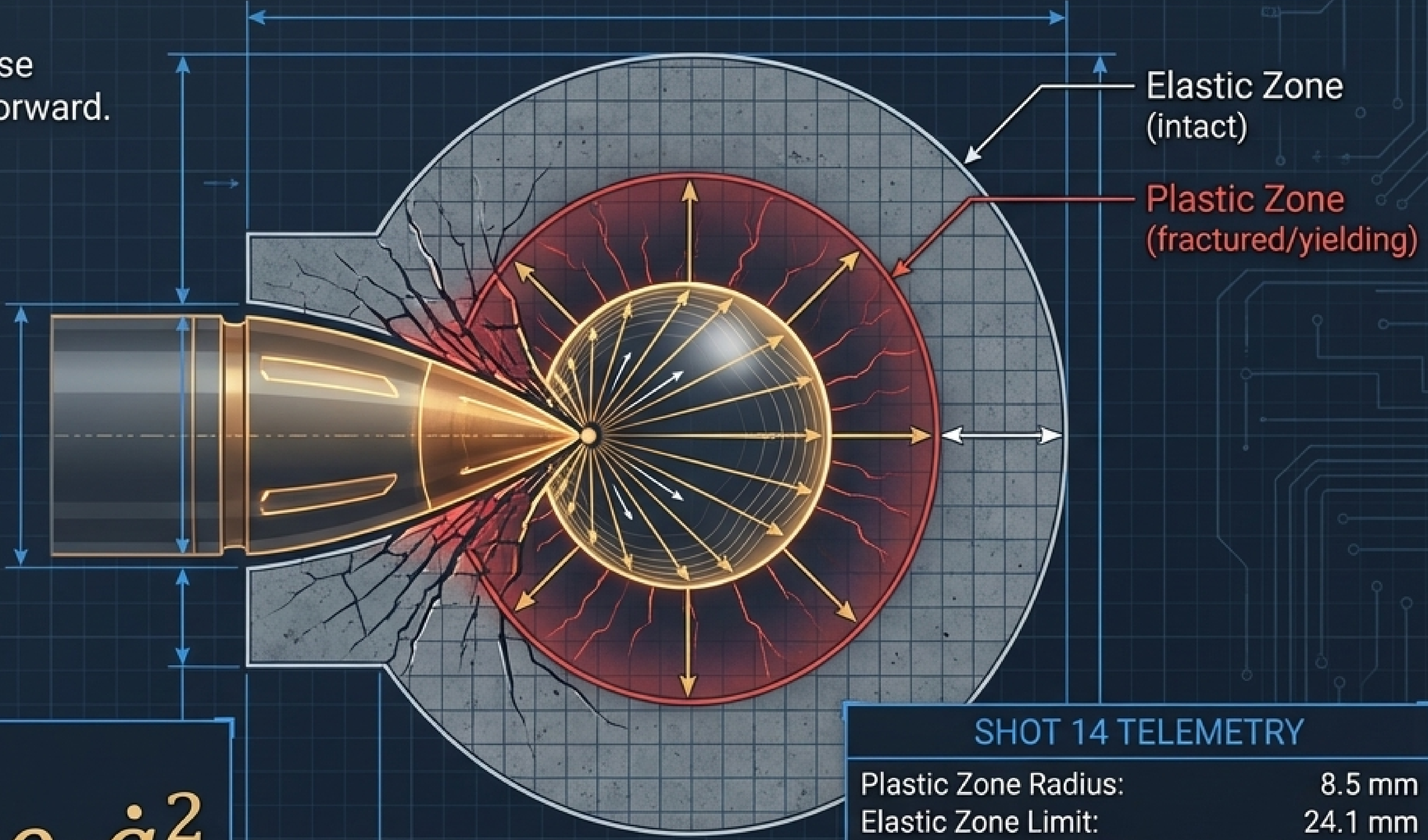
The Origin of Resistance: Cavity Expansion Theory

The Core Concept:
Every infinitesimal element of the nose expands a local cavity as it pushes forward.



$$\dot{a} = V \sin\theta$$

($\dot{a}$ is radial cavity velocity, V is axial projectile velocity, θ is the local nose angle).



The Pressure Equation:

$$p = A\tau_0 + B\rho_c\dot{a}^2$$

Total pressure equals the material's static shear resistance **PLUS** the inertial resistance of the mass being pushed.

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Plastic Zone Radius:	8.5 mm
Elastic Zone Limit:	24.1 mm
Local Expansion Pressure (p):	1,250 MPa
(Stress Red: Static Shear 350 MPa, Ballistic Brass: Inertial 900 MPa)	
Radial Velocity ($\dot{a}$):	142 m/s
Local Nose Angle (θ):	35.2°

Deconstructing the Force: Static vs. Dynamic

$$F = \left(\frac{\pi d^2}{4}\right)(Sf_c) + \frac{\pi d^2}{4}\Big)\left(N^* \rho_c V^2\right)$$

Force Layer Cake:
The inertia of the concrete being violently moved. Highly dependent of velocity. Independent of nose shape.

Layer 2: The Dynamic Bow-Shock ($N^* \rho_c V^2$)
The inertia of the concrete being violently moved. Highly dependent on velocity and nose shape.

Layer 1: The Static Base (Sf_c)
The baseline resistance of the material under confinement. Independent of velocity. Independent of nose shape.

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(Force Analysis at V_0)

Total Force:	~256 kN
Static Force:	240 kN (94%)
Dynamic Force:	16 kN (6%)

Insight: In typical impact regimes (800 m/s), the static strength completely dominates the deceleration profile.

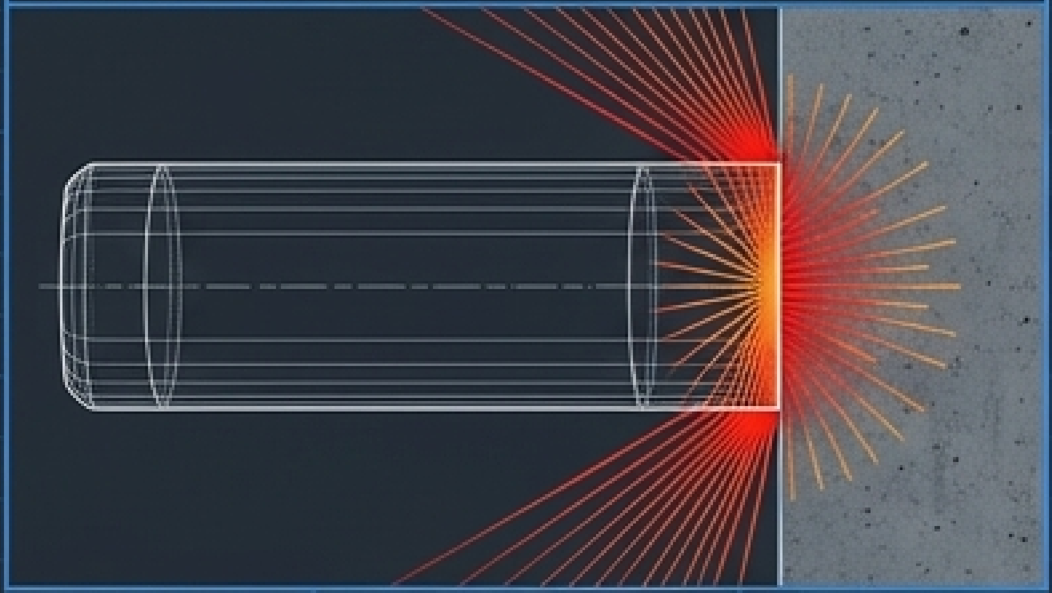
The Nose Factor (N^*): Solving the Geometry Problem

The Concept: The static force ignores geometry. The dynamic force is entirely governed by it. N^* normalizes the dynamic integral across the surface of the nose.

The Integral: $N^* = \dfrac{8}{d^2} \displaystyle\int_0^h \dfrac{y\ y'^3}{1 + y'^2}\, dx$

(Integral of the projected area and local expansion velocity).

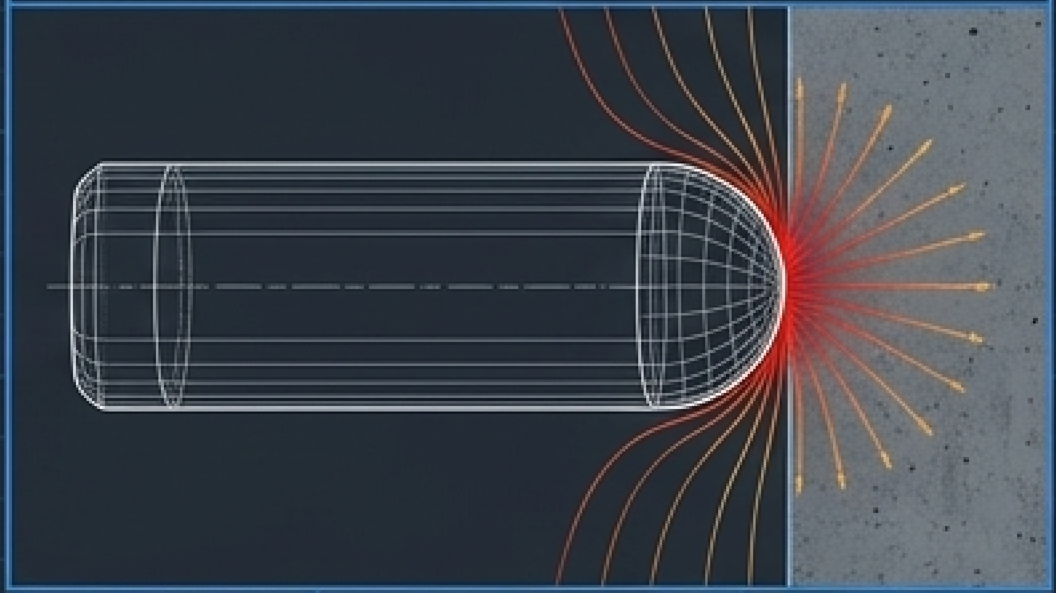
Flat Nose



N* = 1.000

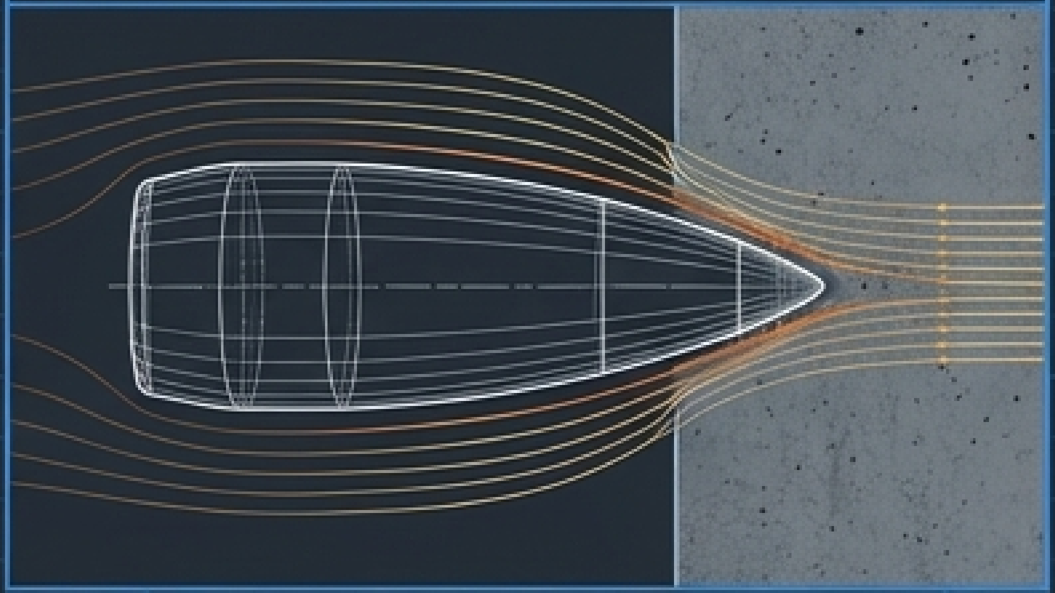
(Blunt impact, massive dynamic penalty).

Hemispherical



N* = 0.500

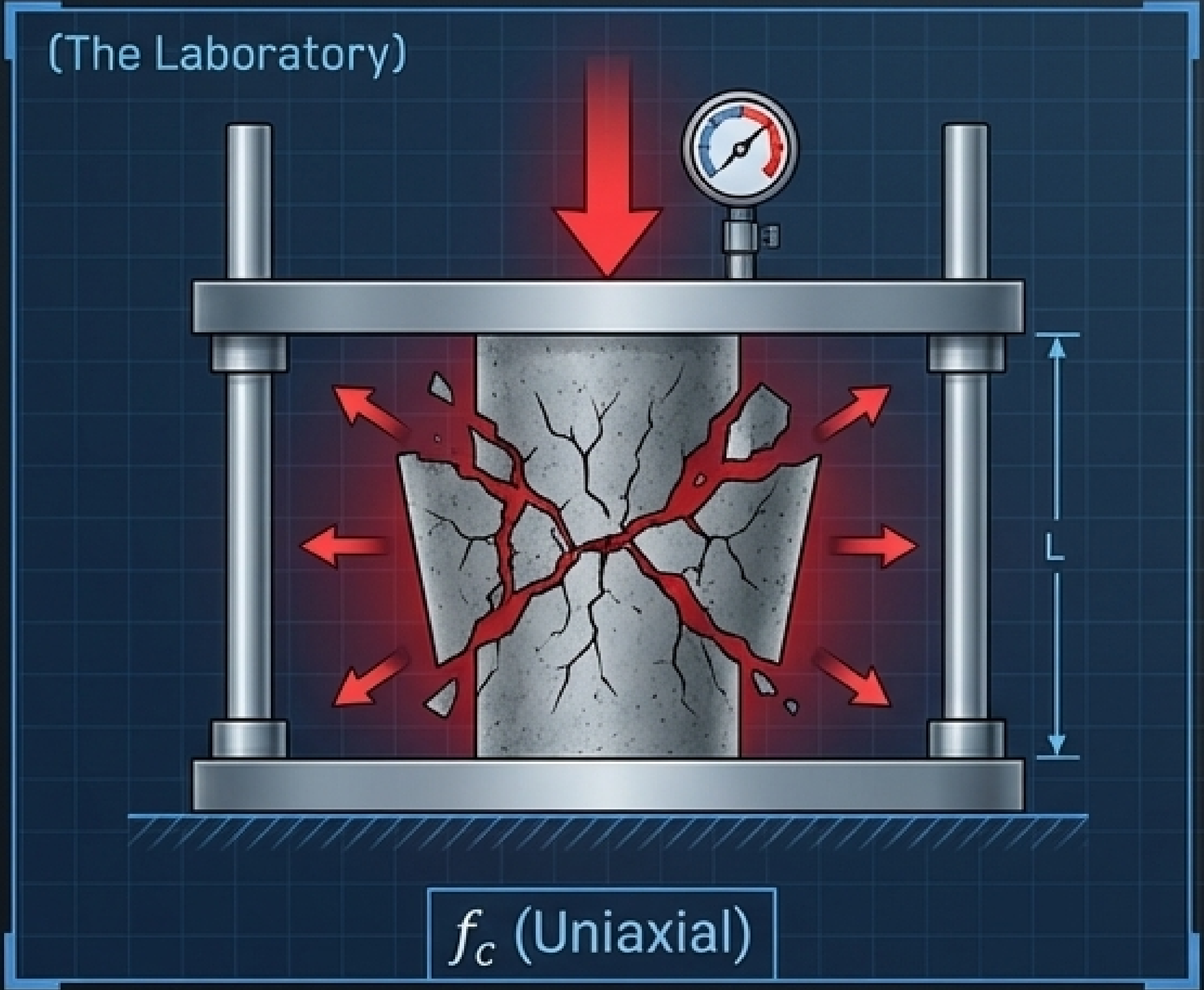
Ogive (CRH=2)

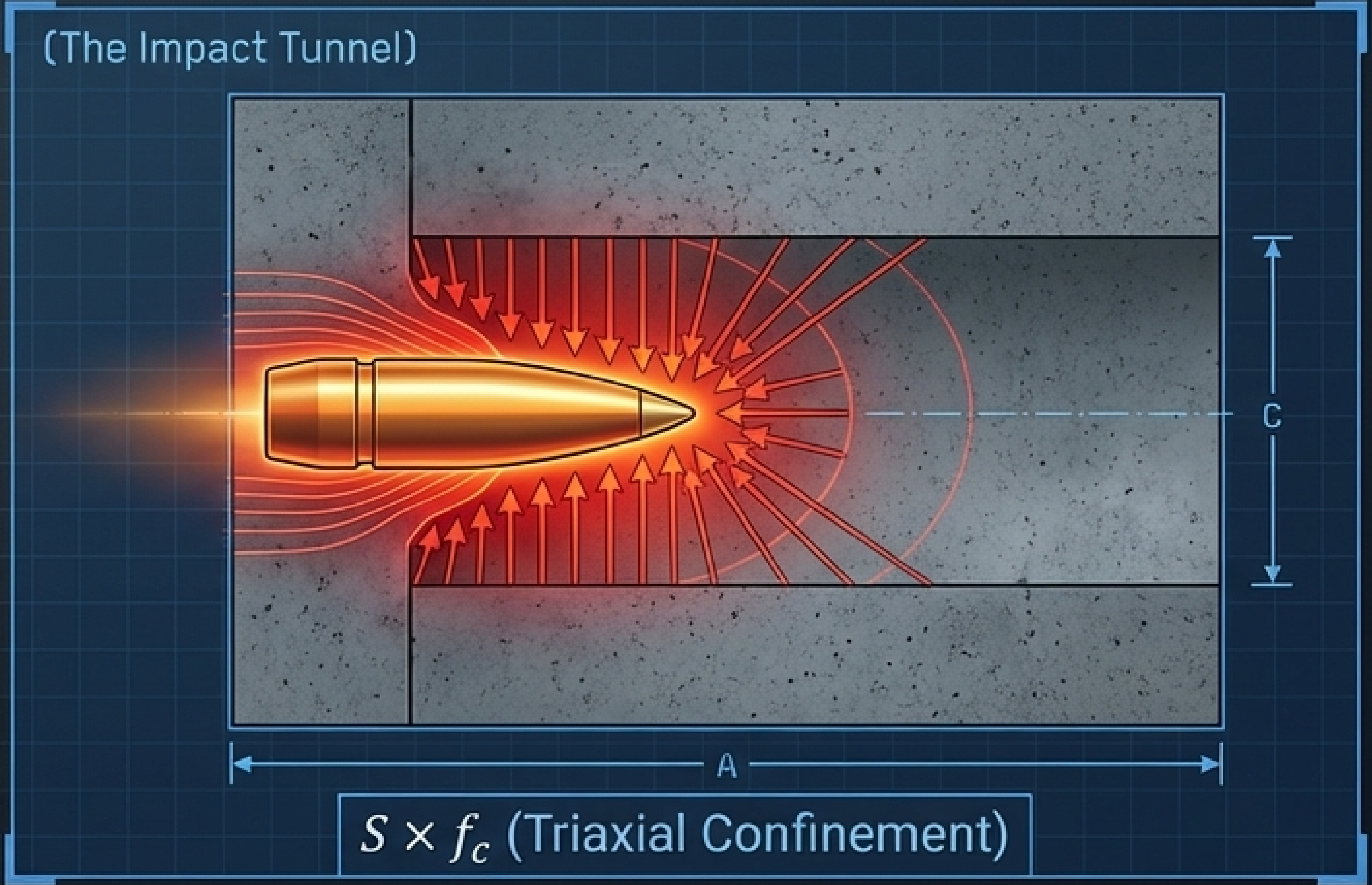


N* = 0.156 (Shot 14)

Takeaway: An Ogive CRH=2 is over 6 times more aerodynamically efficient inside concrete than a flat cylinder, severely reducing dynamic resistance.

The S Parameter: The Confinement Bridge





The Empirical Correlation

Because triaxial dynamic behavior is too complex for closed-form analysis, it is captured empirically:

$$S = 72.0\, f_c^{-0.5}$$

Insight: Concrete is incredibly sensitive to confinement pressure. Under impact, its effective resistance is an order of magnitude higher than lab test values.

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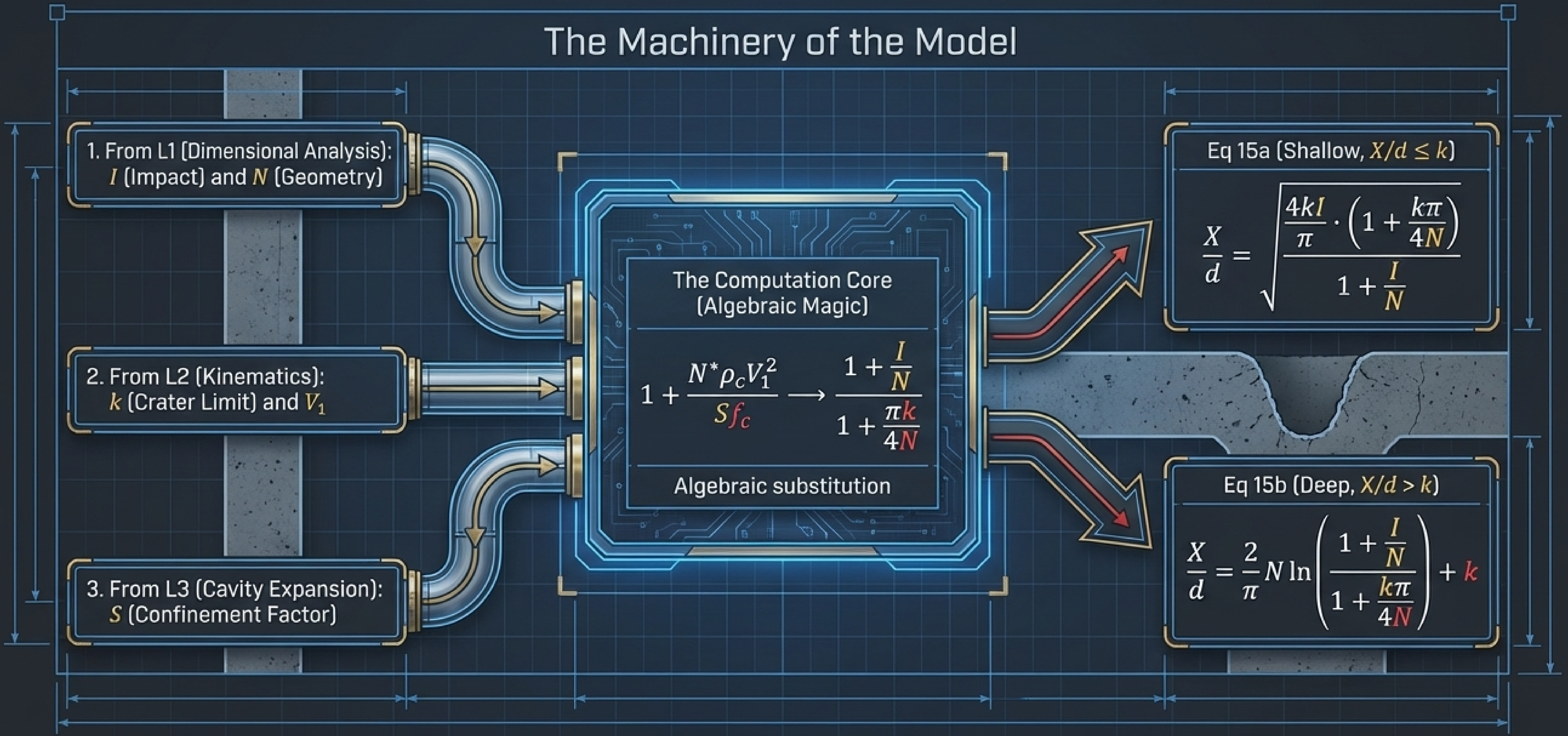
Lab Strength (f_c):
35.2 MPa

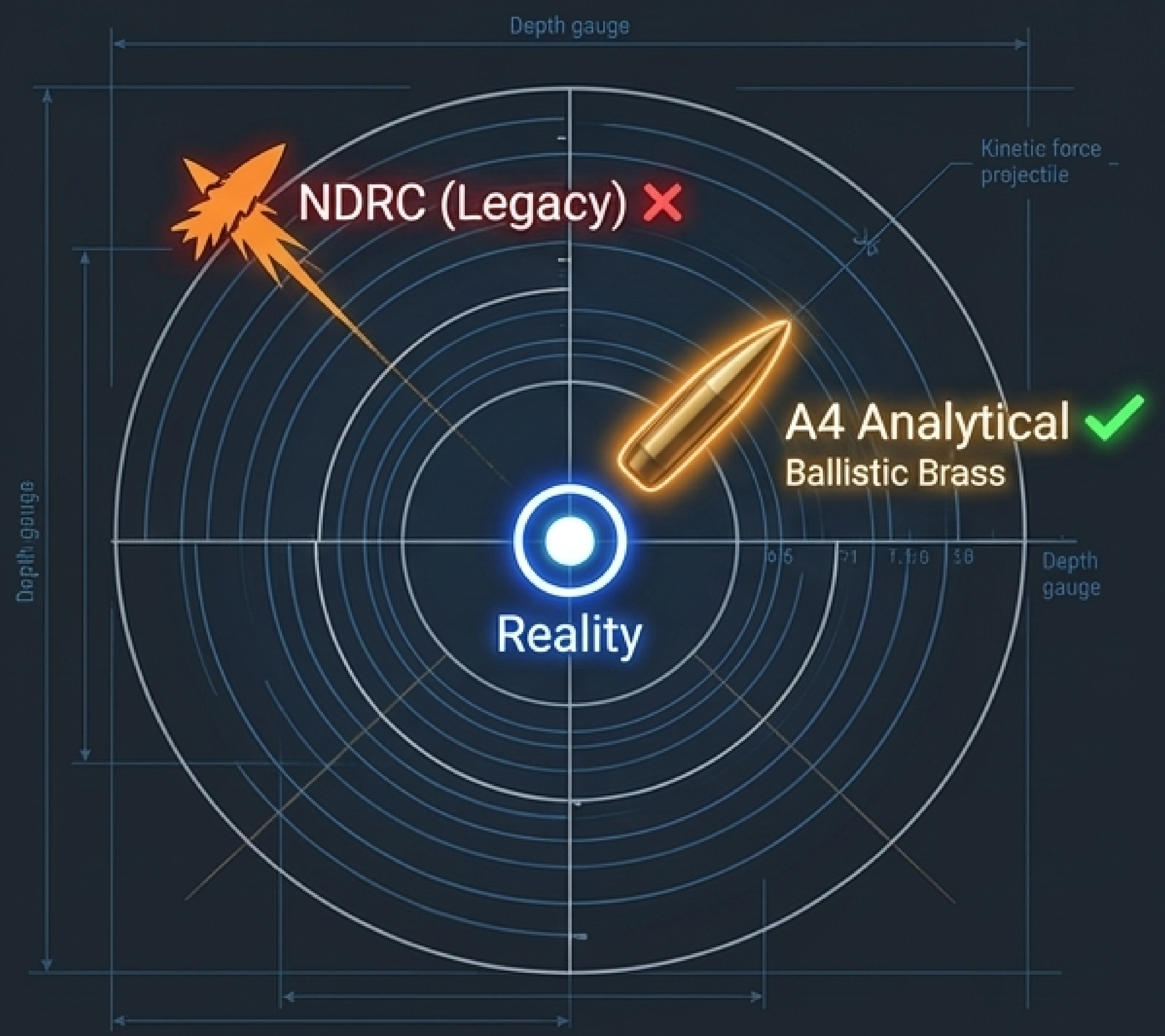
Calculated Confinement Factor (S):
12.1 (Paper uses baseline 12)

Effective Impact Strength ($S \times f_c$):
422.4 MPa

The Equation Engine: Synthesis & Adimensionalization



Proof of Concept: Validation Against Reality



The Final Equation Run (Shot 14):

$$I = 8.455, N = 125.9, k = 2.0$$

$$\frac{X}{d} = \left(\frac{2}{\pi}\right)(125.9)\ln\left(\frac{1.067}{1.012}\right) + 2.0 = 6.217$$

$$X_{anal} = 6.217 \times 26.9 \text{ mm} = 167.2 \text{ mm}$$

Verdict

Verdict	
Physical Test Measurement (X_{test}):	173.0 mm
A4 Analytical Model Error:	3.4% ✓
Legacy NDRC Model Error:	20.5% ✗

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- Lak Strength (fc): 35.2 MPa
- Calculated Confinement Factor (S): 12.1
- Effective Impact Strength (S x fc): 422.4 MPa

From 7 physical variables, to 3 dimensionless groups, through cratering kinematics, bounded by cavity expansion theory—we predict reality within 4 millimeters.

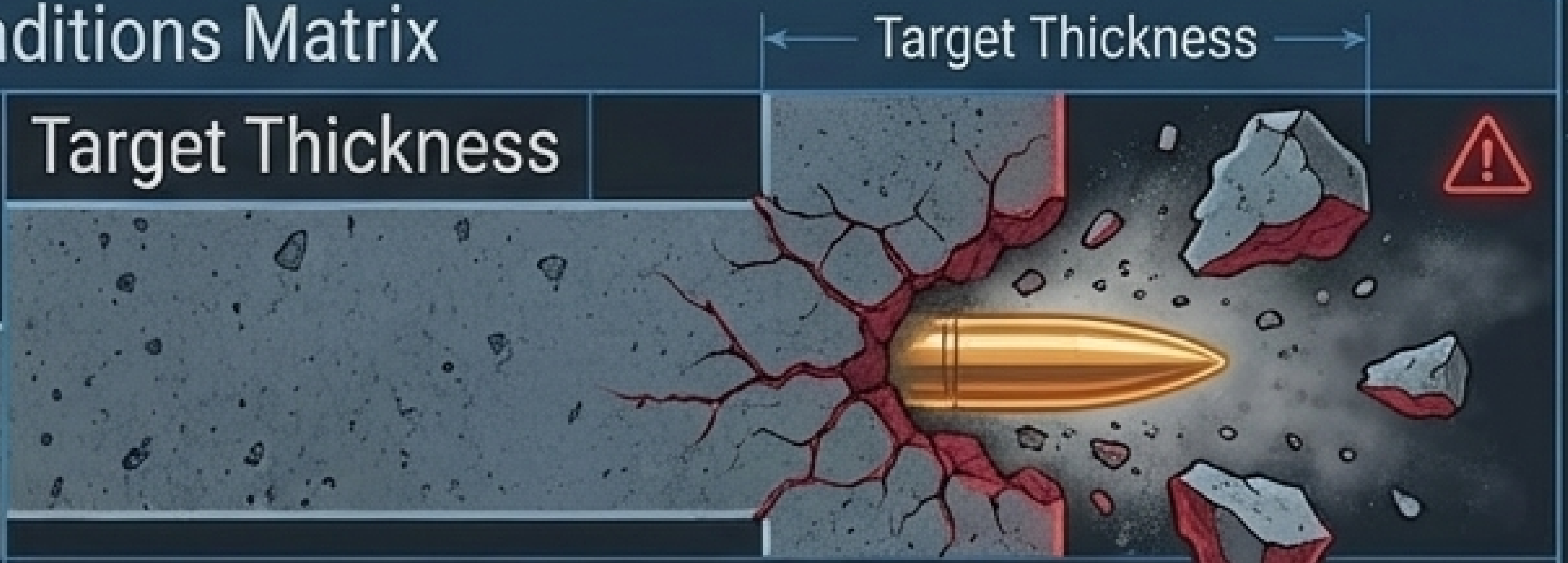
System Boundaries: Limits of Validity

Boundary Conditions Matrix

Projectile Rigidity

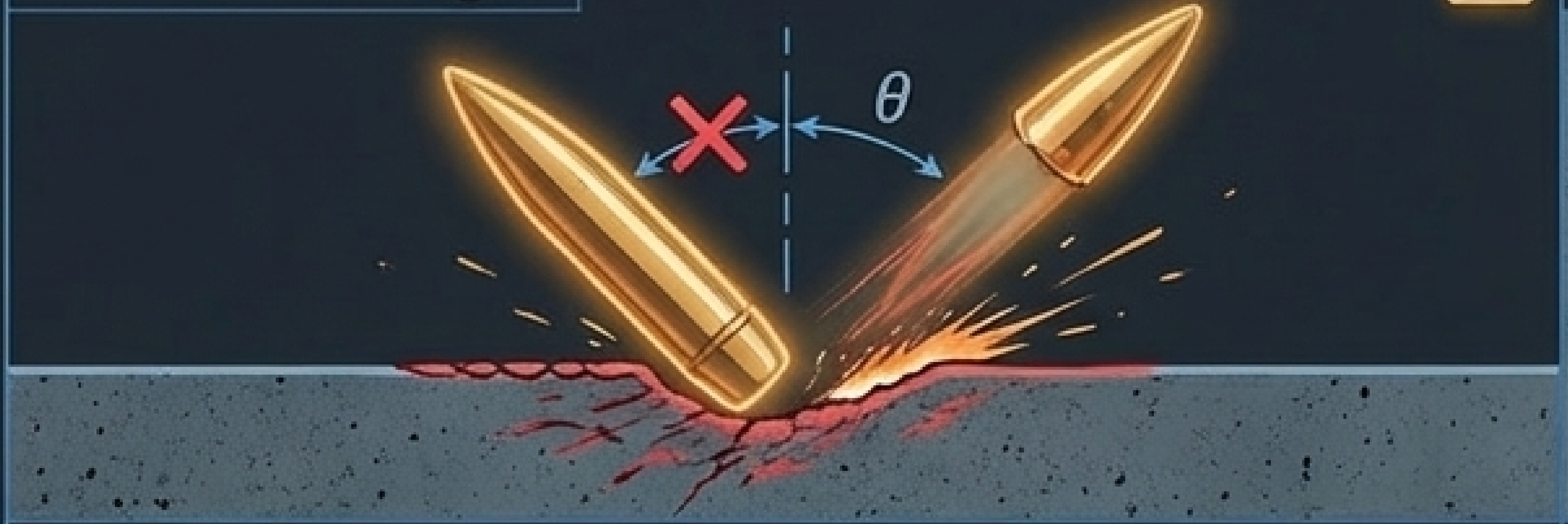
Valid only for non-deformable projectiles (e.g., hardened steel < 800 m/s). If the projectile erodes, the force equation invalidates.

Target Thickness



Valid only for semi-infinite targets. Target thickness must be ≥ 3X to avoid rear boundary effects.

Incidence Angle



Valid only for normal (perpendicular) incidence impacts.

Aggregate Scale

The continuous medium assumption requires bullet diameter to be significantly larger than the aggregate dimension ($d/a \gg 1$).

The A4 Model: Not a black box, but a transparent ballistic blueprint.